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# Students' empirical thinking in solving mathematics problems

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***Abstract***
*The research purpose is to investigate and explore a solution of non-directed of mathematics problems presented visually or algebraic, and to embed the empirical verification thinking. The problems are from Researcher Repertoire, test item of Teacher Profession Education of National Indonesia, and Flanders Mathematics Olympiad. We analyze the students' empirical verification thinking of their solutions, i.e. the trend of the thinking, model of representation, and completeness of the logical steps. The results are: the pattern of thinking tends to linear model or of meta-pattern, the description tends to be non-linear or varies of the solution, and the logical steps tend to be a non-recognizable form of thinking. Our recommendations are that the more visual representations need multiple representations, the algebraic thinking needs more the visual illustrations, and the visual images needed in solving mathematics problems.*

***Keywords: algebraic, empirical verification thinking,image, meta-pattern, representation, visual***

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## INTRODUCTION

The fluency of the verbal descriptions are related only to the term of surfacefeatures of perception or images. Those are mental images builded representational thinking of mathematics problem comprises of algebraic and visual. Rif'at (1998, 2001) founded that geometric objects considered not inherently in-depth means as static features, the visual representation is a model of thinking, understanding problem, and to articulate the examination of perception or imagination. For example, their thinking arrived at the algebraic-analytic solution and, at the same time becoming dull, and disturbing their thinking. It means that the student's visual thinking is holding and looking more accessible or recognizable geometry objects. They give some examples of particular cases recognizable to describe the object's features but can't see the whole situation. In this research, the consideration is to develop in-depth visual and algebraic representations look at the mathematics problems.

This research is about the students' thinking (Rif'at, 2018), closing to mathematics gap, and exploring the solutions to performance from problems. The students use each representation to solve the issues presented visually or need to visualize them in making or arranging the solutions (Darmawan et al., 2020; Rif'at et al., 2019). A review of research on preparing the students' thinking noted that while many pre-service students expect to work in mathematics knowledge, most have little knowledge or experience in the visual. That is, mental imagery needed a model of thinking related to the mathematics representations. But, no priming effect suggested mental images when solving the problems. The researcher observes that perceptual experience can distinguish mental imagery from arithmetic (or algebra thinking).

The research has examined the students' thinking models in solving the problems. That is a different aspect from a theoretical framework where a solution provides insights, provides a comprehensive understanding of the representations. Concerning Landa (1976), this research underlies thinking and performance in solving the problems using the representations, mental operations viewed as a kind of imagery thinking "algebraic" and "visuals." That re-formulated algebraic and geometric representations mentally as cognitive activities analyzed into algebra operation, semi-analytic or visual. The theory of learning specifies taught not only knowledge but the thinking of representation as well. That is how to discover solutions and think on their own. The emphasis is on cognitive operations of the representations which make up models of thinking, particularly by empirical verification. Concerning the representations, proposed some solving strategies based on the models of thinking. That is to recognize the visual and algebraic thinking classified in different situations—the thinking models verified by mapping competencies as depicted in Table 1.

**Table1. A mapping of representations related to thinking models**

| Algebraic representation | Visual representation | Thinking model |
|---|---|---|
| Formulating | Figuring | Transforming |
| Connecting | Constructing | Simplifying |
| Modelling | Modifying | Manipulating |

The students thinking models in the first reaction of a mathematics problem contain visual responses to transform the representations in the solving. But, they used algebra knowledge, an algorithmic or analytic manipulation point of view. The second reaction is to simplify critical attributes of the geometric shapes, i.e., the students connect the visual situation to the algebraic concepts and relating the situation analytically to solve the problems. In mathematics, the critical attributes stem from the definition of the concept (Tsamir & Mandel, 2008) that looks merely memorizing. Visual thinking thought as a phenomenon that could introduce experimentally to a certain extent (Adler & Davis, 2006; Çaylan Ergene & Haser, 2021; Kilhamn & Bråting, 2019; et al., 2018). Rif'at (2017b; 2018, p. 11) ensures that a visual perception based on geometric concepts that operated to mental imagery. The representation supports both algebraic and visual for comprehension and creativity and to improve students' thinking in solving the problems. Sophocleous, *et al*. (2009), report that the visual model in problem-solving facilitates students' comprehension and creates solution-finding opportunities.

Tall (1991) argues that visualization is more effective than conventional approaches in strengthening students' intuitions and facilitating the learning. Tall (1992) considers visualization as a tool that serves to attract students' attention by drawing geometric concepts and models with varying effects to implicate the presence of various mathematical systems and various spaces. That helps students acquire an abstraction and to improve their cognitive independence and productivity, and to ensure meaningful learning and retention of information.

Concerning the two representations thinking, there is an etymological sense of a concept. That is the thinking by formulating a problem and figuring it, connecting concepts and constructing the visual models, and modelling a situation for modifying the relationships. The goals are for determining the available information, abstract or the practical sense. In the thinking models, there is a mind mapping method, a series of abstractions that represented algebraically or visually. That is a relation between operations and the implicit mapping in the logical connection seen similar to individual representation. For example, a statement: if f is any trigonometric functions then $f(x + 2\pi) = f(x)$, there is an implicit mapping that mainly bring students to a visual of sine (or cosine or others) and a translation $2\pi$ to the left. It is not algebraic representation, and difficult to do that.

The students need to deal with simple geometrical representations and concepts rather than an arithmetic operation. From the study, it understood that students need a visual (the simplest) for constructing an equation. They want the equation to get another one according to a problem. For example, transforming the representation to a recognizable one but still not yet brings to the solution. Another algebraic expression arranged by a matrix or transformation of two order matrix. It looks practical, i.e., only taking a point before and after the transformation. That is a linear transformation, and of course, the students come to an incorrect answer. It is a symbolic expression that precedes and leads to the intervention of the solution.

The doing math is to connect algebra and geometry ideas, develop logically and the thinking, and using them in solving the problems.The research problem is "How do students of mathematics problems needed visual representation or presented visually come to understand the relationship between algebra and visual solutions"? That is the development of the Rive' model (Rif'at, 2017a) of understanding representation based on the thinking. For instance, Rittle-Johnson & Star (2009); Star & Seifert (2006) found that the association of geometry understanding and the analytic one developed the sense of spatial perception, and Verschaffel, et al. (2007) considered students' activities enhance their spatial abilities. That is useful to make concepts visible, i.e. in demonstrating the importance of the representations. Star (2005) attribute to prefer the analytical process in teaching (even though they use visualization in their work) since visualization cannot form a proof. It is not easy to establish and understand visual models.

Wu (1999) argues that it is chiefly images that allow one to discern the proof types in mathematics and the ways used to solve a problem, whereas Stigler & Hiebert (1999) state that while the role of images in implicating and understanding relationships is undeniable, images can never be a part of proof on their own and can only implicate the accuracy of a judge. Similarly, some researchers discuss an old prejudice against visualization in mathematics, which concerns the reasons behind the preferences: assuming that mathematics should be exact, analytical, symbolic, and algorithmic.

Particularly, to look at the arguments from the solutions which make the use of visual representation is unavoidable. Different from Flores (1993) that pointed out the use of the figures are more common among educationists, whereas the preliminary study shows that the lecturers and the students tend to do algebraically as a tool in solving problems (included in teaching and learning). The position of Vinner & Hershkowitz (1989) is to explain the reason for rejection of the figures as a radical perception of philosophic belief in which a figure based piece of evidence is neither stable nor valid, and denote the radical belief in the non-visual aspect of mathematics, and this research gives another reason. That is, through the two representations, the students open their thinking to build more strategies in solving problems, although starting from illustrations.

In that consideration, this research is to analyze students' preference of representational thinking, visual or algebraic, and an encouragement to use it in learning mathematics and revealing to the extent to make use of the preferences in the problem-solving process. The visualization and the analytical (or algebraic) preferences in solving the problems thought to support the processes and to understand their models of thinking used by the students and to encourage them to use the preferences empirically.

## METHOD

Data collected five times at 3 different classes, odd semester in academic year 2020/2021. The students are already get mathematics contents in the courses, and the content is school mathematics. The problems are about empirical thinking from algebraic and visual representations. The visual thinking mainly based on the representation that used in solving the problems. The algebraic thinking was based on symbolic manipulation of the problems.

Theresearcher conducted during lecturing of geometry, algebra, trigonometry, and integrated learning course (calculus and real analysis). The focus is on intervention, building-up of using the representation, and exploring the empirical thinking models. That is a meta-pattern categorized by type of representations according to thinking as a kind of performance spectrum in solving mathematics problems.The design is to verify solutions by considering the students' empirical thinking. That recognized according to the solutions steps by the representations. The empirical thinking is of verification, explored from the solution as depicted in Figure 1.

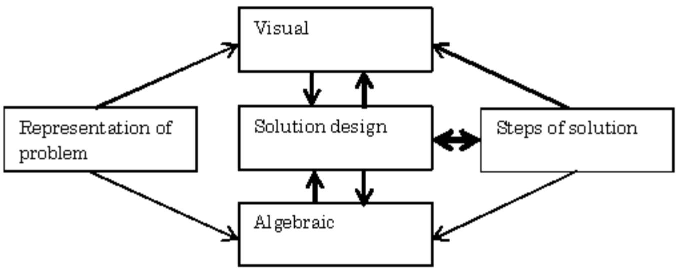


**Figure 1. Diagram of research design**

The research is an experimental design of teaching and learning. The researcher, the students and the lecturers play an active role in the teaching and learning process. The researcher facilitatesthe learning and comprehension to represent the solutions algebraically and or visually. The approach is pedagogical, to gain the researcher'view and idea of the research. The treatment controlled by the observed data trend about the significant of the representation oscillation in the answer. The observation managed through a deep discussion according to the same model of solution. That is an empirical change of mathematical thinking, and the changing of the steps arranged in a pattern that describing a model of thinking.

To construct the models, the researcher separates the representations, make a relation and the thinking model. In algebraic representation, there are symbols, system of equation, changing of the representation, and arousing another equation. The models of the empirical verification comprise of the algebraand of visual thinking.

The growth and the modelling analysed qualitatively for identifying the patternsof the representations. The data was mainly designed on the basis of mathematical representations. To generate data, the representations are qualitative case, which used clinical interventions (project approach), discussion. The intervention and discussion sought to examine the students' experiences of the representation in solving mathematics problems and their views of the thinking context.

In line with the research question "How" do students of mathematics problems needed visual representation or presented visually come to understand the relationship between algebra and visual solutions"?, there are two major categories used to process the data: (1) students' idea about representation thinking prescribed in the solutions and (2) the ways in which the students recognize representational difficulties in the solving. Then, the researcher recorded and analysed the visual and algebra thinking of the representations by verification of the empirical solutions. The models of thinking depicted in Figure 2.

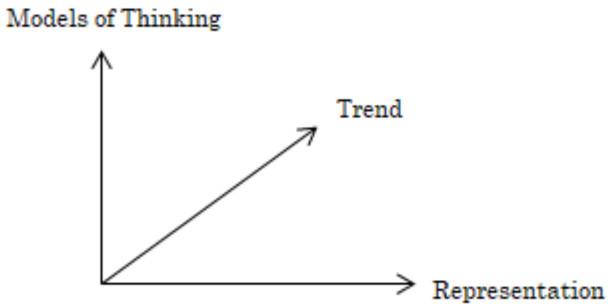


**Figure 2. A Graphical Model of Empirical Verification Thinking**

The representation axis shows the visual or algebraic used by the students and the trend shows how consistence they are in using the representation. The consistency or how big the representation used is a model of solution that described in a graph and show trend. The trend is the model of thinking that describing a meta-cognitive type.

## RESULTS AND DISCUSSION

Problem 1: Proving that, if α and β are two acute angles and α < β, then *sin* α <*sin* β. Twenty students visualize right triangles (13 students) and 7 of them draw any triangle and then construct the heights. And, there 3 students are not to visualize it. The thirteen students use particular measure of the two acute angles, i.e. 30 and 60 degrees for α and β respectively. They give the value of each of the function and the compare it. The students did not count to get the value, but just recalling from the previous learning. That is the solution.Seven students use symbols for taking the ratio of *sin* α and *sin* β, but not used in steps of a solution. They give the ratio and compare it by inequality or analytic, i.e. the ratio is $\frac{t}{x}$ and $\frac{t}{y}$ for *sin* α and *sin* β respectively. Their conclusion is $\frac{t}{x} < \frac{t}{y}$ and then they give the reason that because of $x>y$ by perception. One student makes the same visual, but different logic used in the solution. That is, from the visual $x>y$ so α < β. The student uses the visual, but not to solve the problem. The student firstly describes the two sinus functions by ratio from the visual and then concluded that α < β based on the visual. The solution by contradiction (a type of doing proof) shows that the algebraic representation without the visual look like no guidance. They conclude α > β only because of the equivalence of implication (contra positive), but still not to prove. And, the next step also can't bring to the rationality of logic. That is not a proof, as one of algebraic thinking problem. The logic statement in proof is for describing the equivalence.

Another type of problem 1 is to prove: if α <β and both are two acute angles then $\frac{\sin\alpha}{\alpha} < \frac{\sin\beta}{\beta}$. There are some 'complicated' solutions. There are 18 students solving the problem. During class discussion was understood that all students used algebraic representation because of last experience. The logical connection of the algebraic representation is not a proof of the expression. For example, five students write that, suppose that $\sin\alpha > \sin\beta$ then α > β. Contradiction to the antecedent, so the statement is true.

When the researcher needs the detail and ask the students for explanation, they said: "if $\sin\alpha > \sin\beta$ then in a right triangle shows that α > β." That is a problem of doing proof of algebraic (or analytic) representation. That is an equivalent statement that usually used when the students considered the simplest one for elaborating or proving, but not a contradiction way. An interesting respond is of using visual (geometric shape) representation. They draw a right triangle, i.e. the two angles are in one triangle. In that case, α > β so based on their perception concluded that $\sin\alpha > \sin\beta$. Why do they not use the same proof from the original statement? Most of the students said that one way to prove is by contradiction.

Problem 2: The students ask to find the *tan* of *β* from Figure 3.

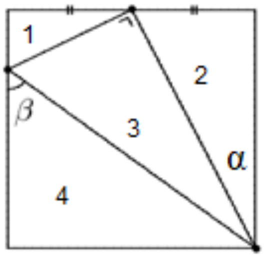


**Figure 3. A square is divided into four rectangular triangles**

There are 14 students participated in solving the problem. Ten students take visual number 4 of the figure to elaborate the situation of the solution. Tree of them work in the original picture. The picture number 4 used by them to solve it. The students get a part of the visual of number 4 and then drawing it outside the original, and complete it as in Figure 4. There is no information of the completed by the measures. From discussion, the students belief that β = 30 + 30 = $60^0$. Their beliefs based on measurement by protractor. They did not directed to ratio of *tan*, but construct other line segments to get magnitude of the angles.

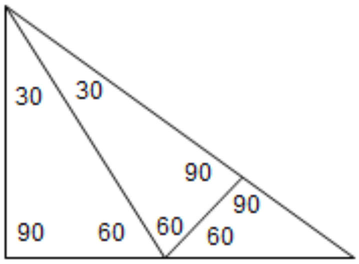


**Figure 4. Part of the original picture constructed by the students**

Working from the original picture, three students respectively state that: β = 180-(30+90) = $60^0$, starting from $\tan \alpha$ to find $\alpha$ by calculator equal to 26.54 but can't get β and *tan β,* and the last one student' answer is $\alpha = \frac{1}{3}$ of right angle and equal to 30 degree. The conclusion is that β = $60^0$.After a discussion or clinical investigation understood that a mind-mapping of the students are to look for particular triangles. That is their experiences during learning. But, that is an image when meeting a visual representation. Other intact students answer the problem algebraically, i.e. using Pythagorean and the practical understanding. In general, there are two types the solutions, but more algebraically than visual empirical verification. They have not yet used visual representation, and not focus on the visual illustration.

Problem 3: Starting with a square of side 1, a regular hexagon is constructed, concentric with the square as in Figure 5. The students ask to find the area of the intersection of the both figures.

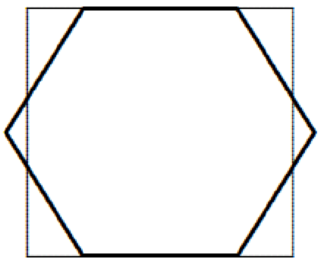

**Figure 5. A square of side one intersects with a hexagon**

There are 37 students of semester 2 become participants in solving the problem. At amount of 31 students start their solutions, using area formula of a hexagon with a variable of the side. Sixteen of the students end their solutions with the variable in square, not find a number. The steps of the answers are full arithmetic works. That is a relation in algebraic but not as well as the representation, means that just simple relation. For examples, area of the intersection is 1.598 $s^2$ where $s$ is the length of the hexagon side (10 students), i.e. the area is equal to area of hexagon minus the square; the intersection area is $A = \frac{3}{2}\sqrt{3}s^2$ and $s = \left(\frac{1-2x}{6}\right)^2$ where $x$ is a variable of the square side not of the hexagon (4 students); and calculating area of one equidistance triangle in the hexagon using *sin function*, multiplied by 6 and get the final answer is $s^2 = 2\sqrt{2}a^2$ where $s$ =

$a$, and the hexagon area, i.e. $\frac{3}{2}a^2\sqrt{3}$ where $a$ is side of the hexagon. Fifteen of other students give answer that the area is $\frac{1}{6}\sqrt{3}$ where the length of hexagon side is 1/3 and without minus the two small figures out-side the square (4 students); negative, i.e. $\frac{4-3\sqrt{3}}{8}$ that is equal to area of square minus the hexagon (1 student); the length of hexagon side is 2/3 and the answer is bigger than 1 (5 students); and the very big number of the answer because the students determined that the hexagon side is 9 (6 students).

Six students give different answers look more complicated by algebraic thinking and the relations in the representation. That is no adding information from the visual but the students give many numbers in the solutions. The intersection area is $\left(\frac{3\sqrt{3}}{2}\right)(1-2a)^2$ where 1- 2$a$ is the length of hexagon side; divide the visual into 2 trapezoids and a rectangular; using diagonals and conclude that the area of hexagon is equal to area of the square; using Pythagorean to get the hexagon side by the equation $a^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{a}{2}\right)^2$, and the inter-section area is $\frac{\sqrt{3}}{2}$; and count the area of 6 triangles outside the intersection using assumption that the two of the outside square is equal to the two triangles inside the square, so the intersection area is $x^2 - ab$, where $x$ is side of the square, $a$ and $b$ respectively are the right side of the four triangles inside the square.

All of the answers are algebraic representations in their relations without any logic in the visual situation. There are some visuals made by the students, but not in relation to the question. They can work in arithmetic skills but the visual look like for information of the algebraic thinking. The visuals are two different visual made by the students. The intersection is not right and some others of the students put the hexagon inside the square, and determined 6 triangles for getting an answer.

Problem 4: The students ask to find the cosine of the top angle $\alpha$ of one of the lateral faces as in Figure 6.

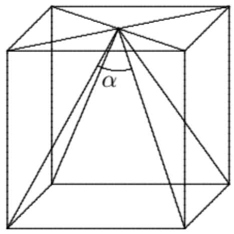


**Figure 6. A cube with a constructed regular pyramid**

There are 18 students of semester 5 solved the problem. Seven students change the visual as depicted in Figure 7.

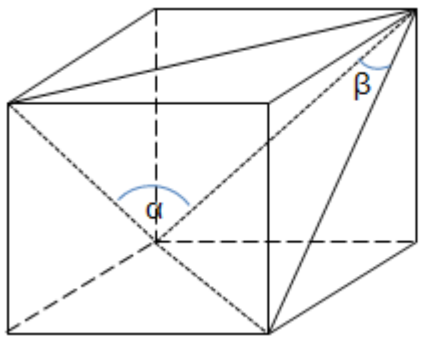


**Figure 7. New construction after understanding problem situation**

After constructing the figure, six students write the angle α at Figure 6 is same as at Figure 7. It looks no relation, but the students think that triangle at Figure 7 is equidistance, so the angle is 60 degrees. The students try to bring the problem to the area of the triangle to get α. One student redraws Figure 7 without diagonals of the top plane. She draws a net of pyramid inside the cube and concludes that four triangles of the net are equidistance, so α is 60 degrees. Eight students take the pyramid out of the cube and think it. The results assign to the models of thinking within each cell of Table 2.

**Table 2. Distribution of problems**

| Problem | Type of Information | Sum |
|---|---|---|
| 1 | Need a visual/picture | 23 |
| 2 | Visual representation | 14 |
| 3 | Geometry knowledge | 37 |
| 4 | Spatial | 18 |

The researcher used a classifying construction as depicted in Table 3.

**Table 3. The performance of the representations based on thinking models**

| Visual Representation | Algebraic Representation | Thinking Models |
|---|---|---|
| Based on the visual | Using the formula | Transforming |
| Algebraically | Separated from the algebra | Simplifying/Formulating |
| Alternately | Changed to the different one | Processing |
| Algebraically | Visual manipulation | Manipulating |
| Using the algebra | Visually | Percepting |
| Algebra algorithmic | Developed to the algebraic | Completing/Connecting |

According to the degree of the representations used in the steps of the solutions, the flow is of the students' empirical thinking to complete the solution or getting an answer. The empirical thinking verified from the solutions and short class discussion during the research.The visual and algebra representations that used are mainly by formulas. The students use a formula in geometry and then algebra. The algebra manipulation is from the relation but without the visual. That is a model of simplification. In the discussion, the students say that the visual representation helps them to memorize the formula and then solve it algebraically. In the process of the solution, the students back to the visual representation, to separate the algebra manipulation for getting another relation in the representation and then working algebraically. That is a type the thinking process.

The consistency of the thinking looks at an effort to get another visual built from the original used alternately in the solution. The students draw some visuals added to the original, but the solution forward divergently styles. It looks at the same adequate thinking between the two representations for a solution. The completion of the students' thinking is for the algebra relation. When they face the variables of an equation, the students try to get more visual representation to complete a comparison to the system. In the visual image, the students solve the problems algebraically.

Perception coloured on the students' solution. Visual perception is for getting a solution but recognizable previously. They consider the visual representation to get algebra relations that possibly solved. So, in the algebraic representation, the students' perception is visually but using algebra in the solution. That is the insight into the steps of the solutions based on algebra. They develop their thinking patterns to the algebra and the algorithmic.

**Visual Thinking**

The thinking models correspond to the solution of a problem, based on pictures manipulated and constructed toward the solution or affects the way searching for 'eureka solution.' That is step by step linear thinking where the geometry knowledge interconnected in 'solution space' (Presmeg, 2006). The thinking synthesizedan intuitive to process the representation inductively, and to use geometric concept image by constructing the facts for manipulating. The thinking designed for a step-by-step visual representation and combined as well as in getting an idea to answer the problems. The process is a hierarchical mapping with not many algebraic relationships to their imagery. Fischbein (1977) state that the students can create a mental image of a concept and see how the information fits with what they already know, and their learning permanent.

When the students look at a picture, they were able to process the information fast, but not in the solution such posing at the picture. They start from the visualization to get any algebraic relation and never see again the visual representation (Rif'at, 2019, p. 8). Another useful indicator of visual thinking from the representation is as another performance needed in the learning. Many tasks oriented to a computation or calculation, but, for enhancing the thinking, the students need a visual construction as an illustration or helping them to solve the problem.

The difficulty of solving problems by the visual representation and the thinking is in area of cognition. That is to interchange the visual representation to the algebraic and vice-verse. The reason is that school mathematics thought need the visual representations when solving the problems (Darmawan et al., 2021; Yamaguti, 1993). A good consideration is that the ability has to maintain the model of thinking, for detecting the learning difficulties. The assumption is the mathematics behavior could be developed in cycling creation

(Rif'at, 2001, p.101), i.e., a model of visual thinking.Visual thinking is faster than algebraic, but the last used in a solution. It is a texture of solving a mathematics problem by the students; also, lecturers, as said by some of them. So, the visual representation not always for the algebraic expression, it is also developed in many mathematics problems.

**Algebraic Thinking**

The analytic or symbolic expression often associated to a formal proof received by the students. But, in teaching and learning, that must produce a manageable material to exclude the learning constraints of the students gradually. This study discovered that at least 85% of solving the problems presented algebraically or analytically. Mundy & Lauten (1994) founded that "students often come to the error when solving problems algebraically."Lecturers also based them-self on the algebraic expression more and more. They demonstrate the representation to keep 'mathematical' understanding.

The algebraic thinking is linearly, but holistically in the system and interconnected. The students look for the right one at their disposal. Their thinking responded more because of the connection-they see many paths, visually or algebraically, to differing answers and adding information to decide which representation to take to the answer. In teaching and learning mathematics, it must be an optional representational between the visual and algebraic thinking. Students need to consider that the visual representation also measured, and the possibility to judge the algebraic in solving a problem. That comes to the assumption of thinking in domain of representation. It is often convenient to approximate the representations by the thinking in solving the problems.

In a crypt-analysis of algebra thinking, the evidence combined with the visual cases. That is powerful when solving a problem. That is the algebraic representation and the thinking evaluated by quasi-utility or a visual manipulation as an epistemic syntax. The expectation is a concept of representation for building constructive solutions. Dienes (1960) states that the concept is an entropy expected that the algebraic representation concerning to another representation-because, many algebraic representations depend explicitly on logical relationships, and sometimes controversial in mind.

The choice of algebraic representation shows an easy to check the truth and to draw the decisions. So, to encourage students making accurate solutions to algebra is to pose as much as the same idea independently of the different representations. The thinking tends divergent but not understands having the right answer. The attributes of the thinking are sequential but often veer into unusual and different trajectories (Dreyfus, 1991). The solution is illogical, or no directed conclusions; the students view a problem leading to breakthroughs representations.

Generally, the students' thinking path is by representing, manipulating and converting any problem to algebra expressions and the relationships. Students had not enough visual information to change the representation for showing 'the same' illustration. The students' image mainly on the visual representation but the solution is analytic-symbolic. The path depicted in Figure 8.

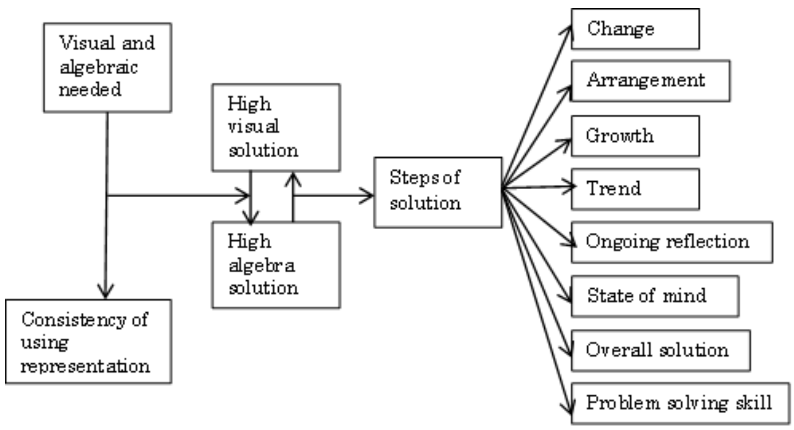


**Figure 8. The path of thinking of mathematics problems**

The path of the problem solutions observed on four phases of thinking that verified on students' empirical performance. The models of thinking analysed in eight 'keywords' based on steps of the solutions, i.e., change, arrangement, growth, trend, ongoing reflection, state of mind, overall solution, and the problem-solving skill. The steps appeared primarily on the students' choices of algebra or visual solutions. That is the basis of the steps, although alternately depends on imagination and the difficulty. The choices are in accordance to a problem situation that could be constructed by the students' mind for solving the problems. That is still in the visual representation or needed an algebraic relationship.

The meta-patterns of the thinking look at the keyword, although not ordered in the solutions. For example, after changing a representation to another one, the students arrange steps but not on the first representation that selected. The arrangements of the solutions look hard to understand because of mixing the representations in every step. So, the growth to the solution has no particular trend, but an ongoing that mainly based on the simple one, i.e., algebraically.

The overall solutions are symbolic-analytic, while the visual unused as the solving imagination. The visual representation also not to be completed, so the algebraic expression conducted by the symbolic manipulation. But, there is also the usage of the visual when the students face complicated steps for solving a problem. For example, when trying to solve a problem that if $\alpha < \beta$ are two angles in the first quadrant then $\sin \alpha < \sin \beta$ the students visualize the trigonometric relation at one or two triangles. At a particular triangle, they change the position of the angles, and at the same two triangles are to change the name of the angles. The conclusion drew accorded to the proportion, i.e. which one bigger based on the visual.

## CONCLUSION

The connectivity of the thinking models brings the representations to the meta-cognitive dimensions. One of research result is a visual representation is also a combination of thinking, and the algebraic is tends to be routine. The combination is a construct of empirical thinking as a strategy to solve the problem and to explore the situation. While algebraic thinking was based on symbol manipulation or thinking analytically, the strategy is often less of meaning. That does not diminish the need for the development of mathematical thinking, but rather it encompasses them alongside other vital dimensions such as attitudes and dispositions. The students see mathematics as an integral part of the representation as a broader identity and helped define their etymological sense of a solution. Their skills are more than a structure of information, and they try to pass into the solutions. The researcher suggested for having and displaying the representations related to thinking activity in learning mathematics. That is an affirmation of the fact (abstract or analogy image) to know the strong relationship based on the thinking in solving mathematics problems. In this sense, the relationship is more than just a social connection and includes pedagogical approaches. There has been much written about the characteristics and nature of quality pedagogical relationships between teacher and student, but here we want to highlight the importance of the student – teacher connection in building students' mathematical identity. We suggested that effective teachers are able to connect with both student and subject, and in the process they facilitate the students' relationships with the subject – their mathematical identity.

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