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PRESERVING SUBINJECTIVITY DOMAIN OF A MODULE

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Abstract
An R-module M is said to be indigent if its subinjectivity domain consists of only an injective module. In this paper, we study some properties of the indigent module. We give some examples of rings which have an indigent module. We also prove that subinjectivity domain of a module is preserved and reflected under equivalence.

Keywords: *equivalence, indigent module, subinjectivity domain*

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INTRODUCTION

Let R be a ring. An R-module U is said to be Y-injective if and only if, for every submodule X of Y, any homomorphism $f: X \to U$ extent to a homomorphism $g: Y \to U$. For an R-module U, the injectivity domain of U is defined as the collection of all R-modules Y such that U is Y-injective, and is denoted as $\mathfrak{In}^{-1}(U)$. If $\mathfrak{In}^{-1}(U)$ is the smallest, U is called a poor module. The condition of $\mathfrak{In}^{-1}(U)$ to be the smallest is equivalent to having $\mathfrak{In}^{-1}(U)$ consists of only a semi-simple module (Alahmadi, 2010). As an alternative perspective of this concept, in (Pinar, 2011) the concept of subinjectivity is introduced. An R-module M is said to be K-subinjective if for every R-module L, with K is submodule of L, any homomorphism $f: K \to M$ extent to a homomorphism $g: L \to M$. For an R-module M, the collection of all R-modules K such that M is K-subinjective is called subinjectivity domain of M, and is denoted as $\underline{\mathfrak{In}}^{-1}(M)$. If $\underline{\mathfrak{In}}^{-1}(M)$ is the smallest, M is called an indigent module. According to (Pinar, 2011), the condition of $\underline{\mathfrak{In}}^{-1}(M)$ to be the smallest is equivalent to having $\underline{\mathfrak{In}}^{-1}(M)$ consists of only injective modules. The existence of poor module over any ring was proved in (Noyan, 2011). However, the existence of indigent module over arbitrary ring remains an open problem till now.

As an attempt to solve the open problem, in this paper, we are able to prove that the subinjectivity domain of a module is preserved by an equivalence. As a consequence, the property of a module being indigent is also preserved by the equivalence. So, if a ring R has an indigent module, any ring S that equivalent to R also has an indigent module. The important result of these two theorems is that the ring $M_n(\mathbb{Z})$ has an indigent module.

Throughout this paper, a ring R is an associative ring with 1, and $J(R)$ denotes its Jacobson's radical. For an R-module M, $E(M)$ denotes its injective envelope. The category of all R-modules over ring R will be denoted by R-Mod.

Definition 1. (Pinar, 2011) Let U and X be R-modules. U is said to be X-subinjective, if for every R-module V with $U \leq V$ and for every homomorphism $\nu: X \to U$, there exists an homomorphism $\varphi: V \to U$ such that $\nu = \varphi \circ \varepsilon$, where $\varepsilon: X \to V$ is an inclusion (i.e ν extent to φ, in notation $\varphi|_X = \nu$).

Theorem 1. (Pinar, 2011) Let U and X be R-modules. We have the following equivalent assertions:

1. U is X-subinjective.
2. For every homomorphism $f: X \to U$ and for any essential extension V of X, there exists a homomorphism $g: V \to X$ such that $g|_X = f$.
3. For every homomorphism $f: X \to U$, there exists a homomorphism $g: E(X) \to U$ such that $g|_X = f$.

The subinjectivity domain of an R-module M is defined as $\underline{\mathfrak{In}}^{-1}(X) = \{U \in R - Mod \mid U\ is\ X - subinjective\}$

RESULTS AND DISCUSSION

We begin this section by the formal definition of the indigent module and some of its elementary properties.

Definition 2. (Pinar, A. 2011) An R-module X of whose the subinjectivity domain is the intersection of subinjectivity domain of all modules over R is called indigent module. In other words, X is indigent if $\underline{\mathfrak{In}}^{-1}(X) = \bigcap_{\mathrm{K \in R-Mod}} \underline{\mathfrak{In}}^{-1}(K)$

Theorem 2. (Pinar, A. 2011) An R-module M is indigent if and only if $\underline{\mathfrak{In}}^{-1}(M) = \{\mathrm{N \in R - Mod \mid N\ is\ injective}\}$.

Using Theorem 1 and Theorem 2, we can characterize an injective module as follows

Theorem 3. Let M be an indigent R-module. Let N_1, N_2 be R-modules, and $\varphi: N_1 \to N_2$ is a monomorphism. Consider the map $\varphi^*: Hom_R(N_2, M) \to Hom_R(N_1, M)$ given by $\varphi^*(g) = g \circ \varphi$. The module N_1 is injective if and only if φ^* is surjective.

Proof.

($\Leftarrow$)

Let $f \in Hom_R(N_1, M)$. Since φ^* is surjective, there exists $g \in Hom_R(N_2, M)$ such that $f = \varphi^*(g) = g \circ \varphi$. Since φ is a monomorphism, we conclude that $N_1 \in \underline{\mathfrak{In}}^{-1}(M)$, and since M is indigent, we have N_1 is injective.

($\Rightarrow$)

Let $f \in Hom_R(N_1, M)$. Since N_1 is injective, the exact sequence $0 \to N_1 \xrightarrow{\varphi} N_2$ is split. Hence there exist a homomorphism $\psi: N_2 \to N_1$ such that $\psi \circ \varphi = id_{N_1}$. Define $g = f \circ \psi$, we have $\varphi^*(g) = g \circ \varphi = (f \circ \psi) \circ \varphi = f$. Hence φ^* is surjective.

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As a consequence of **Proposition 2.4 and Proposition 3.2** in (Pinar, 2011), we have the following result

Theorem 4. Let $\{M_x\}_{\mathrm{X}}$ be a family of modules indexed by an index set X. Let M_x be an indigent module for some $x \in \mathrm{X}$. Then the following assertions, hold

1. $\prod_{x \in \mathrm{X}} M_x$ is indigent
2. $\bigoplus_{x \in \mathrm{X}} M_x$ is indigent

Proof.

1. Let $T \in \underline{\mathfrak{In}}^{-1}(\prod_{x \in \mathrm{X}} M_x)$. Since $\underline{\mathfrak{In}}^{-1}(\prod_{x \in \mathrm{X}} M_x) = \bigcap_{x \in X} M_x$ (Pinar, 2011), then $T \in \underline{\mathfrak{In}}^{-1}(M_\lambda)$, and thus T is injective since M_x is indigent. We conclude that $\prod_{x \in \mathrm{X}} M_x$ is indigent module.
2. The similar way can be applied as the proof of 1.

■

The following result can be found in (Pinar, A. 2011), here we give the detailed proof.

Theorem 5 $\oplus_{q\ is\ prime}\mathbb{Z}_q$ is an indigent $\mathbb{Z}$-module

Proof.

Observe that any nontrivial simple Abelian group G is finite and cyclic, hence it is isomorphic to $\mathbb{Z}_p$ for some prime number p. We will show that G is divisible if and only if G has no maximal subgroup. If G is not divisible, then, there exists a prime number q such that $qG \subset G$. Hence $G\ /\ qG$ is a vector space over $\mathbb{Z}_q$. Let $X = \{x_i\}_{i\in I}$ be its basis, for some index set I. Take some $x_j \in X$, then $span\{x_k, k \neq j\}$ is maximal subspace of $G\ /\ qG$. Therefore, by correspondence theorem, $span\{x_i, i \neq j\} = H\ /\ qG$ for some subgroup H of G containing qG. Since $H\ /\ qG$ is maximal subgroup of $G\ /\ qG$, H is maximal in G, a contradiction. We conclude that, G is divisible. Conversely, let G be divisible, and suppose that G has maximal subgroup, say H. Since H is maximal, $G\ /\ H$ is simple, and hence isomorphic to $\mathbb{Z}_q$ for some prime number q, which implies that $qG \subseteq H$. On the other hand, since G is divisible, then $qG = G$, and hence $G = qG \subseteq H$, a contradiction. So G has no maximal subgroup. Since divisible group is equivalent to injective $\mathbb{Z}$-module, we conclude that any $\mathbb{Z}$-module G is injective if and only if G has no maximal submodule, equivalently, if $Rad(G) = G$. By **Proposition 4.4** in (Pinar, 2011), $\oplus_{q\ is\ prime}\mathbb{Z}_q$ is indigent module.

■

A ring T with the property that for any ideal I, J of T, either $I \subseteq J$ or $J \subseteq I$, is called a chain (uniserial) ring. If a ring U is direct sum of chain rings, we called U as serial ring. In (Pinar, 2011), if U is an Artinian serial ring and $J(U)^2 = 0$, then U has an indigent module, namely $U/J(U)$. Also, over Artinian chain ring, the noninjective module M is indigent. We thus give the following examples:

Example 1. Let p, q be two distinct prime numbers. Consider the ring $S = \mathbb{Z}_{p^2}\oplus\mathbb{Z}_{q^2}$. Clearly, S is Artinian, and since both $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_{q^2}$ are chain ring, then S is Artinian serial ring. We have $J(S) = J(\mathbb{Z}_{p^2})\oplus J(\mathbb{Z}_{q^2}) = \langle p\rangle\oplus\langle q\rangle$. Let $x = (pu, qv) \in J(S)$, then $x^2 = (p^2u, q^2v) = (0,0)$. Hence $J(S)^2 = 0$. So, the module $S/J(S) = \mathbb{Z}_{p^2}\oplus\mathbb{Z}_{q^2}\ /\ \mathbb{Z}_{pq}$ is an indigent module.

Example 2. For any prime number q and natural number n, as a module over itself, any submodule of $\mathbb{Z}_q$ is an indigent module.

For the final part of this paper, before we give the main result of our work, we need some preliminary definition and lemmas, as follows.

Definition 3. (Wisbauer, 1991) Let R, S be rings. A functor $F: R - Mod \to S - Mod$ is called equivalence if there exists a functor $G: S - Mod \to R - Mod$ together with a pair of natural isomorphisms $\mu: G \circ F \to id_{R-Mod}$ and $\eta: F \circ G \to id_{S-Mod}$, where id_{R-Mod} and id_{S-Mod} are identity functor of $R - Mod$ and $S - Mod$ respectively. The ring R is thus said to be Morita equivalent to ring S.

Clearly, if F is an equivalence, then G is also an equivalence. In this case, G is called invers (equivalence) of F and vice versa.

Lemma 1. (Wisbauer, 1991) A functor $F: R - Mod \to S - Mod$ is an equivalence if and only if it is Fullyfaithful and representative.

Lemma 2. (Wisbauer, 1991) Given an equivalence $H: R - Mod \to S - Mod$. For any R-module M, N and homomorphism $f: M \to N$, the following statements hold:
1. f is a monomorphism if and only if $H(f): H(M) \to H(N)$ is a monomorphism
2. $H(E(M)) = E(H(M))$

Theorem 6 Let R and S be two rings, and $H: R - Mod \to S - Mod$ be an equivalence. Let U, V be R-modules. The module U is V-subinjective if and only if $H(U)$ is $H(V)$-subinjective.

Proof.

Let U be a V-subinjective R-module. Let $\varphi': H(V) \to H(U)$ be a homomorphism, and $\varepsilon': H(V) \to W'$ is a monomorphism, where W' is any extension of $H(V)$. Since H is representative, there exists $W \in R - Mod$ such that $H(W) \cong W'$. Suppose $\delta: W' \to H(W)$ is the isomorphism, then $\delta \circ \varepsilon'$ is a monomorphism. Since H is full, there exist a homomorphism $\varphi: V \to U$ and a homomorphism $\varepsilon: V \to W$ with $H(\varphi) = \varphi'$ and $H(\varepsilon) = \delta \circ \varepsilon'$, and H being an equivalence. It guarantees that ε is also a monomorphism. Since U is V-subinjective, there exist $\nu: K \to A$ such that $\varphi = \nu \circ \varepsilon$. By defining $\nu' = H(\nu) \circ \delta$, we have $H(\varphi) = G(\nu \circ \varepsilon) = G(\nu) \circ G(\varepsilon) = G(\nu) \circ (\delta \circ \varepsilon') = (H(\nu) \circ \delta) \circ \varepsilon' = \nu'\varepsilon'$. We conclude that $H(U)$ is $H(V)$-subinjective S-module.

Conversely, let $H(U)$ be $H(V)$-subinjective S-module. Given any homomorphism $\nu: V \to U$ and monomorphism $\varepsilon: V \to W$. Since H is equivalence, $H(\varepsilon)$ is monomorphism. Since $H(U)$ is $H(V)$-subinjective, there exists $\varphi': H(W) \to H(U)$ such that $\varphi' \circ H(\varepsilon) = H(\nu)$. Now, since H is full, there exists a homomorphism $\varphi: W \to M$ such that $H(\nu) = \varphi'$. We have $H(\varphi \circ \varepsilon) = H(\nu) \circ H(\varepsilon) = H(\nu)$. By the faithful property of H, we conclude that $\varphi \circ \varepsilon = \nu$. Hence U is V-subinjective.

■

Theorem 7. Let R and S be rings, and $F: R - Mod \to S - Mod$ be an equivalence. An R-module A is indigent if and only if $F(A)$ is indigent S-module.

Proof.

Let A be an indigent module. Let Y be an S-module such that $F(A)$ is Y-subinjective. Since F is representative, there exists an R-module X such that $F(X) \cong Y$. By Theorem 6, A is X-subinjective. Since A is indigent, then X is injective. By Lemma 2, we have $Y \cong F(X)$ is injective. Thus $G(A)$ is indigent.

For the converse, let $G: S - Mod \to R - Mod$ be the inverse of F. Since $A \cong H(G(A))$, we have the desired result.

■

As the direct consequence of Theorem 7, we have the following results

Corollary 1. The ring of all $s \times s$ matrices over $\mathbb{Z}$, where s is natural number, have an indigent module.

Corollary 2. The ring of all $q \times q$ matrices over $\mathbb{Z}_{t^n}$, with t is prime and q, n are natural numbers, have an indigent module.

CLOSING REMARKS

As a closing remark, according to the result, the equivalence functor preserves and also reflects subinjectivity domain of a module. Moreover, as a consequence, an equivalence is also preserves and reflects indigent module. For further study, it is interesting to study another type of functor that preserves and reflects subinjectivity domain of a module that is not an equivalence. Also, since in an arbitrary category we know the concept of injective object, the study of indigent object will be an interesting topic in category theory research area.

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